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Linearized Oscillations for Even-Order Neutral Differential Equations

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We are concerned with the oscillation of all bounded solutions of some even-order linear neutral delay differential equations. We establish a comparison theorem and a linearized oscillation result, and we prove the existence of a bounded positive solution.

1. INTRODUCTION

In this paper we are concerned with the oscillation of all bounded solutions of some even-order linear neutral delay differential equations. In Section 3 we present a comparison theorem, in Section 4 we establish a linearized oscillation result, and in Section 5 we prove the existence of a bounded positive solution. Similar results for odd-order neutral delay differential equations have been obtained in [1–6].

Consider the nth-order neutral delay differential equation (NDDE for short)

$$\frac{d^n}{dt^n}[y(t)-P(t)\,G(y(t-\tau))]+Q(t)\,H(y(t-\sigma))=0, \tag{1}$$

where n is a positive integer,

$$P, Q\in C[[t_0, \infty), \mathbb{R}], \qquad G, H\in C[\mathbb{R}, \mathbb{R}], \qquad \text{and} \qquad \tau, \sigma\in \mathbb{R}^+.$$

Let $\rho=\max\{\tau, \sigma\}$. By a *solution* of Eq. (1) we mean a function $y\in C[[t_1-\rho, \infty), \mathbb{R}]$, for some $t_1\geqslant t_0$, such that $[y(t)-P(t)\,G(y(t-\tau))]$ is n times continuously differentiable on $[t_1, \infty)$ and such that Eq. (1) is satisfied for $t\geqslant t_1$.

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Let $t_1 \geqslant t_0$ be a given initial point, let $\phi \in C[[t_1-\rho, t_1], \mathbb{R}]$ be a given initial function, and let z_k, $k=0, 1, ..., n-1$, be given initial constants. By using the method of steps one can see that Eq. (1) has a unique solution $y \in C[[t_1-\rho, \infty), \mathbb{R}]$ such that

$$y(t)=\phi(t) \qquad \text{for} \quad t \in [t_1-\rho, t_1]$$

and

$$\frac{d^k}{dt^k}[y(t)-P(t)\, G(\phi(t-\tau))]_{t=t_1}=z_k \qquad \text{for} \quad k=0, 1, ..., n-1.$$

As usual, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In the sequel, for convenience, we will assume that inequalities about values of functions are satisfied for all large t.

2. Some Basic Lemmas

In this section we present some basic lemmas which will be used throughout this paper.

The first two Lemmas are extracted from [1].

Lemma 1 ([1]). *Let $F, G, P\colon [t_0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that*

$$F(t)=G(t)-P(t)\, G(t-c), \qquad t \geqslant t_0+\max\{0, c\}. \tag{2}$$

Assume that for some positive constant p

$$0 \leqslant P(t) \leqslant p \qquad \textit{for} \quad t \geqslant t_0$$

and

$$G(t)>0 \qquad \textit{for} \quad t \geqslant t_0, \liminf_{t \to \infty} G(t)=0,$$

and that

$$\lim_{t \to \infty} F(t)=L \in \mathbb{R} \textit{ exists.}$$

Then $L=0$.

Lemma 2 ([1]). *Let $F, G, P \in C[[t_0, \infty), \mathbb{R}]$ and $c \in (0, \infty)$ be such that* (2) *holds. Assume that*

$$F(t)>0 \qquad \textit{and} \qquad G(t)>0 \qquad \textit{for} \quad t \geqslant t_0 \textit{ and } \lim_{t \to \infty} F(t)=0.$$

Suppose also that there exists a $p \in [0, 1)$ *such that*

$$0 \leqslant P(t) \leqslant p < 1 \qquad \textit{for} \quad t \geqslant t_0.$$

Then

$$\lim_{t \to \infty} G(t) = 0.$$

Next we will establish the following result.

LEMMA 3. *Consider the* NDDE

$$\frac{d^n}{dt^n} [y(t) - P(t)\, y(t-\tau)] - Q(t)\, y(t-\sigma) = 0, \tag{3}$$

where n is a positive integer,

$$P, Q \in C[[t_0, \infty), \mathbb{R}^+] \qquad \textit{and} \qquad \tau, \sigma \in \mathbb{R}^+. \tag{4}$$

Assume that there exists a positive constant p such that

$$0 \leqslant P(t) \leqslant p \qquad \textit{for} \quad t \geqslant t_0 \tag{5}$$

and that

$$\int_{t_0}^{\infty} Q(s)\, ds = \infty. \tag{6}$$

Let $y(t)$ *be an eventually positive and bounded solution of Eq.* (3) *and set*

$$z(t) = y(t) - P(t)\, y(t-\tau).$$

Then eventually

$$z^{(n)}(t) \geqslant 0, \qquad (-1)^i\, z^{(n-i)}(t) > 0 \qquad \textit{for} \quad i = 1, 2, ..., n \tag{7}$$

and

$$\lim_{t \to \infty} z^{(i)}(t) = 0 \qquad \textit{for} \quad i = 0, 1, ..., n-1. \tag{8}$$

Proof. From Eq. (3) we have

$$z^{(n)}(t) = Q(t)\, y(t-\sigma) \geqslant 0 \tag{9}$$

and because $y(t)$ and $P(t)$ are bounded it follows that

$$\lim_{t \to \infty} z^{(n-1)}(t) = l \in \mathbb{R} \text{ exists.} \tag{10}$$

Hence for each $i=0, 1, \ldots, n-1$, $z^{(i)}(t)$ is monotonic and so

$$\lim_{t\to\infty} z(t)=\gamma \in \mathbb{R} \text{ exists.}$$

We claim that $\gamma=0$. To this end, by integrating both sides of (9) from t_1 to t and then by letting $t\to\infty$ we obtain

$$l-z^{(n-1)}(t_1)=\int_{t_1}^{\infty} Q(s)\, y(s-\sigma)\, ds.$$

This, in view of (6), implies that

$$\liminf_{t\to\infty} y(t)=0.$$

Then by Lemma 1 we get $\gamma=0$. From this and the monotonic nature of $z^{(i)}(t)$ it is easy to see that consecutive derivatives of $z(t)$ alternate in sign; that is, (7) holds. It is now clear that (8) also holds, and the proof is complete. ∎

The next lemma is extracted from [5].

LEMMA 4 ([5]). *Consider the neutral differential equation*

$$\frac{d^n}{dt^n}[y(t)-py(t-\tau)]-qy(t-\sigma)=0, \tag{11}$$

where p, q, τ, and σ are all real numbers. Then every bounded solution of Eq. (11) *oscillates if and only if its characteristic equation*

$$\lambda^n-p\lambda^n e^{-\lambda\tau}-qe^{-\lambda\sigma}=0$$

has no real roots on the interval $(-\infty, 0]$.

The following result is interesting in its own right.

LEMMA 5. *Assume that*

$$p, q, \tau \in (0, \infty) \qquad \textit{and} \qquad \sigma\in[0,\infty)$$

and suppose that every bounded solution of Eq. (11) *oscillates. Then there exists an $\varepsilon_0>0$ such that for every $\varepsilon\in[0, \varepsilon_0]$ every bounded solution of the differential equation*

$$\frac{dn}{dt^n}[y(t)-(p-\varepsilon)\, y(t-\tau)]-(q-\varepsilon)\, y(t-\sigma)=0 \tag{11$'$}$$

also oscillates.

Proof. By Lemma 4, the hypothesis that every bounded solution of Eq. (11) oscillates implies that the characteristic equation of Eq. (11),

$$F(\lambda)=\lambda^n-p\lambda^n e^{-\lambda\tau}-qe^{-\lambda\sigma},$$

has no roots in $(-\infty, 0]$. As $F(0)=-q<0$, it follows that

$$F(\lambda)<0 \qquad \text{for any} \quad \lambda\in(-\infty, 0].$$

Clearly $F(-\infty)=-\infty$ and so

$$M=\max_{\lambda\in(-\infty,0]} F(\lambda)$$

exists and is negative. Therefore,

$$\lambda^n-p\lambda^n e^{-\lambda\tau}-qe^{-\lambda\sigma}\leqslant M, \qquad \text{for} \quad \lambda\in(-\infty, 0].$$

Set

$$\delta=\tfrac{1}{2}\min\{p, q\} \qquad \text{and} \qquad G(\lambda)=\delta(-\lambda^n e^{-\lambda\tau}-e^{-\lambda\sigma}).$$

Then

$$\lim_{\lambda\to-\infty}[F(\lambda)-G(\lambda)]=\lim_{\lambda\to-\infty}[\lambda^n-(p-\delta)\lambda^n e^{-\lambda\tau}-(q-\delta)e^{-\lambda\sigma}]=-\infty.$$

Hence, there exists a $\lambda_0<0$ such that

$$F(\lambda)-G(\lambda)<0 \qquad \text{for} \quad \lambda\leqslant\lambda_0.$$

Let

$$\eta=\max_{[\lambda_0,0]}(\lambda^n e^{-\lambda\tau}+e^{-\lambda\sigma})$$

and set

$$\varepsilon_0=\min\left\{-\frac{M}{2\eta}, \delta\right\}.$$

To complete the proof, it suffices to show that for every $\varepsilon\in[0, \varepsilon_0]$ the characteristic equation

$$\lambda^n-(p-\varepsilon)\lambda^n e^{-\lambda\tau}-(q-\varepsilon)e^{-\lambda\sigma}=0$$

of Eq. (11′) has no real roots in $(-\infty, 0]$. In fact for $\lambda\leqslant\lambda_0$ and because n is even,

$$\begin{aligned}\lambda^n-(p-\varepsilon)\lambda^n e^{-\lambda\tau}-(q-\varepsilon)e^{-\lambda\sigma}&\leqslant\lambda^n-(p-\delta)\lambda^n e^{-\lambda\tau}-(q-\delta)e^{-\lambda\sigma}\\&=F(\lambda)-G(\lambda)<0.\end{aligned}$$

On the other hand for $\lambda_0 < \lambda \leqslant 0$,

$$\begin{aligned}\lambda^n - (p-\varepsilon)\,\lambda^n e^{-\lambda\tau} - (q-\varepsilon)\,e^{-\lambda\sigma} &= F(\lambda) + \varepsilon(\lambda^n e^{-\lambda\tau} + e^{-\lambda\sigma})\\ &\leqslant M + \varepsilon\eta\\ &\leqslant M - M/2\\ &< 0.\end{aligned}$$

The proof is complete. ∎

The proof of the next lemma can be obtained by a slight modification in the proof of Lemma 2 in [4] and will be omitted.

LEMMA 6. *Assume that* $n \geqslant 1$ *is a positive integer,*

$$F, G \in C[[t_0, \infty), \mathbb{R}^+],\ H \in C[\mathbb{R}^+, \mathbb{R}^+],\ \tau \in (0, \infty),\ \sigma \in [0, \infty),$$

$H(u)$ *is nondecreasing,* $H(u) > 0$ *for* $u > 0$ *in a neighborhood of the origin, and that either*

$$F(t) > 0 \qquad \text{for} \quad t \geqslant T$$

or

$$\sigma > 0 \qquad \text{and} \qquad G(t) \not\equiv 0 \text{ on any interval of length } \sigma.$$

Let $m = \max\{\tau, \sigma\}$ *and suppose that the integral inequality*

$$\begin{aligned}&F(t)\,z(t-\tau) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty G(s)\,H(z(s-\sigma))\,ds\,ds_1 \cdots ds_{n-1}\\ &\quad \leqslant z(t), \qquad t \geqslant T\end{aligned}$$

has a continuous positive solution $z\colon [T-m, \infty) \to (0, \infty)$ *such that*

$$\lim_{t\to\infty} z(t) = 0.$$

Then there exists a continuous positive solution $x\colon [T-m, \infty) \to (0, \infty)$ *of the corresponding integral equation*

$$\begin{aligned}&F(t)\,x(t-\tau) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty G(s)\,H(x(s-\sigma))\\ &\quad \times ds\,ds_1 \cdots ds_{n-1} = x(t), \qquad t \geqslant T\end{aligned}$$

and

$$\lim_{t\to\infty} x(t) = 0.$$

3. A Comparison Theorem for NDDEs

The main result in this section is the following comparison theorem for neutral delay differential equations of even order.

Consider the NDDEs

$$\frac{d^n}{dt^n}[y(t)-P_1(t)\,y(t-\tau_1)]-Q_1(t)\,y(t-\sigma_1)=0 \tag{12}$$

and

$$\frac{d^n}{dt^n}[y(t)-P_2(t)\,y(t-\tau_2)]-Q_2(t)\,y(t-\sigma_2)=0, \tag{13}$$

where n is an even integer and

$$\begin{aligned} &P_1\in C^1[[t_0,\infty),\mathbb{R}^+], \qquad P_2, Q_1, Q_2\in C[[t_0,\infty),\mathbb{R}^+],\\ &\text{and} \qquad \tau_1,\tau_2,\sigma_1,\sigma_2\in\mathbb{R}^+. \end{aligned} \tag{14}$$

THEOREM 1. *Assume that* (14) *holds,*

$$\begin{aligned} &P_1, P_2 \text{ are bounded, } P_1'(t)\geqslant 0\\ &\qquad \text{and} \qquad Q_2(t)>0 \qquad \text{for} \quad t\geqslant t_0, \end{aligned} \tag{15}$$

$$\begin{aligned} &\tau_1\leqslant\tau_2 \qquad \text{and} \qquad \sigma_1\leqslant\sigma_2,\\ &P_1(t)\leqslant P_2(t-\sigma_2)\,\frac{Q_2(t)}{Q_2(t-\tau_2)}\\ &\qquad \text{and} \qquad Q_1(t)\leqslant Q_2(t) \qquad \text{for} \quad t\geqslant t_0, \end{aligned} \tag{16}$$

$$\int_{t_0}^{\infty} Q_2(t)\,dt=\infty, \tag{17}$$

and that

if $P_1(t)\equiv 0$ *then* $Q_1(t)\not\equiv 0$ *on any interval of length* σ_1.

Suppose also that every bounded solution of Eq. (12) *oscillates. Then every bounded solution of Eq.* (13) *also oscillates.*

Proof. Assume, for the sake of contradiction, that Eq. (13) has a bounded and eventually positive solution $y(t)$. Set

$$z(t)=y(t)-P_2(t)\,y(t-\tau_2). \tag{18}$$

Then by Lemma 3,

$$z^{(n)}(t) \geqslant 0, \qquad (-1)^i z^{(n-i)}(t) > 0 \qquad \text{for} \quad i = 1, ..., n \tag{19}$$

and

$$\lim_{t \to \infty} z^{(i)}(t) = 0 \qquad \text{for} \quad i = 0, 1, ..., n-1. \tag{20}$$

It also follows by direct substitution that $z(t)$ satisfies

$$z^{(n)}(t) - P_2(t-\sigma_2) \frac{Q_2(t)}{Q_2(t-\tau_2)} z^{(n)}(t-\tau_2) - Q_2(t) z(t-\sigma_2) = 0. \tag{21}$$

By using (16) and the fact that eventually $z^{(n)}(t) \geqslant 0$ and $z(t) > 0$, Eq. (21) yields the inequality

$$z^{(n)}(t) - P_1(t) z^{(n)}(t-\tau_2) - Q_2(t) z(t-\sigma_2) \geqslant 0 \tag{22}$$

for t sufficiently large, say for $t \geqslant T$. We also choose T so large that (19) holds for $t \geqslant T - \max\{\tau_2, \sigma_2\}$. By integrating (22) from t to ∞ we find

$$\begin{aligned} &-z^{(n-1)}(t) - \int_t^\infty P_1(s) z^{(n)}(s-\tau_2)\, ds \\ &\quad - \int_t^\infty Q_1(s) z(s-\sigma_2)\, ds \geqslant 0, \qquad t \geqslant T. \end{aligned}$$

Then by integrating by parts the first integral and by using (15), (16), and the monotonic character of $z^{(i)}(t)$ we obtain the inequality

$$\begin{aligned} &-z^{(n-1)}(t) + P_1(t) z^{(n-1)}(t-\tau_1) \\ &\quad - \int_t^\infty Q_1(s) z(s-\sigma_1)\, ds \geqslant 0, \qquad t \geqslant T. \end{aligned}$$

By repeating the same procedure n times and by noting the fact that n is even we are led to the inequality

$$\begin{aligned} &P_1(t) z(t-\tau_1) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty Q_1(s) z(s-\sigma_1)\, ds \\ &\quad \leqslant z(t), \qquad t \geqslant T. \end{aligned}$$

As all the hypotheses of Lemma 6 are satisfied, it follows that

$$\begin{aligned} &P_1(t) x(t-\tau_1) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty Q_1(s) \\ &\quad \times x(s-\sigma_1)\, ds\, ds_1 \cdots ds_{n-1} = x(t), \qquad t \geqslant T \end{aligned}$$

and so also Eq. (12) has a bounded positive solution. This contradicts the hypothesis that every bounded solution of Eq. (12) oscillates and the proof is complete. ▮

4. Linearized Oscillations

In this section we establish a linearized oscillation result for neutral delay differential equations of even order.

Consider the NDDE

$$\frac{d^n}{dt^n}[x(t)-p(t)\,g(x(t-\tau))]-q(t)\,h(x(t-\sigma))=0, \tag{23}$$

where n is an even integer,

$$\begin{aligned} &p, q \in C[[t_0, \infty), \mathbb{R}^+], \qquad g, h \in C[\mathbb{R}, \mathbb{R}], \\ &\tau \in (0, \infty), \qquad \text{and} \qquad \sigma \in [0, \infty). \end{aligned} \tag{24}$$

Theorem 2. *Assume that* (24) *holds,*

$$\begin{aligned} &\limsup_{t\to\infty} p(t)=P_0 \in (0,1), \qquad \liminf_{t\to\infty} p(t)=p_0 \in (0,1), \\ &\lim_{t\to\infty} q(t)=q_0 \in (0,\infty), \end{aligned} \tag{25}$$

$$0 \leqslant \frac{g(u)}{u} \leqslant 1 \qquad \textit{for} \quad u \neq 0, \lim_{u\to 0}\frac{g(u)}{u}=1, \tag{26}$$

$$uh(u)>0 \qquad \textit{for} \quad u \neq 0 \textit{ and} \lim_{u\to 0}\frac{h(u)}{u}=1. \tag{27}$$

Suppose that every bounded solution of the linearized equation

$$\frac{d^n}{dt^n}[y(t)-p_0\,y(t-\tau)]-q_0\,y(t-\sigma)=0 \tag{28}$$

oscillates. Then every bounded solution of Eq. (23) *also oscillates.*

Proof. Assume, for the sake of contradiction, that Eq. (23) has a bounded nonoscillatory solution $x(t)$. We will assume that $x(t)$ is eventually positive. The case where $x(t)$ is eventually negative is similar and will be omitted. Set

$$z(t)=x(t)-p(t)\,g(x(t-\tau)).$$

Then

$$z^{(n)}(t)=q(t)\,h(x(t-\sigma)). \tag{29}$$

Since $x(t)$ is bounded, $z(t)$ is also bounded. Then by (29) it follows that

$$\lim_{t\to\infty} z^{(n-1)}(t)=l\in\mathbb{R} \text{ exists.}$$

By integrating both sides of (29) from t_1 to ∞, for t_1 sufficiently large, we obtain

$$l-z^{(n-1)}(t_1)=\int_{t_1}^{\infty} q(s)\,h(x(s-\sigma))\,ds$$

which, in view of (25), (27), and the boundedness of $x(t)$, implies that

$$\liminf_{t\to\infty} x(t)=0.$$

Then by an argument similar so that in Theorem 1 in [3] we obtain

$$\lim_{t\to\infty} z^{(i)}(t)=0 \qquad \text{for} \quad i=0,1,2,\ldots,n-1, \tag{30}$$

$$z^{(n)}(t)\geqslant 0, \qquad z^{(n-1)}(t)<0,\ldots,z(t)>0 \tag{31}$$

and

$$\lim_{t\to\infty} x(t)=0. \tag{32}$$

Next we rewrite Eq. (23) in the form

$$\frac{d^n}{dt^n}[x(t)-P(t)\,x(t-\tau)]-Q(t)\,x(t-\sigma)=0,$$

where for t sufficiently large,

$$P(t)=p(t)\,\frac{g(x(t-\tau))}{x(t-\tau)} \qquad \text{and} \qquad Q(t)=q(t)\,\frac{h(x(t-\sigma))}{x(t-\sigma)}.$$

Note also that

$$\liminf_{t\to\infty} P(t)=p_0 \qquad \text{and} \qquad \lim_{t\to\infty} Q(t)=q_0.$$

It is easy to see by direct substitution that, for t sufficient large, $z(t)$ is a solution of the neutral equation

$$z^{(n)}(t)-P(t-\sigma)\,\frac{Q(t)}{Q(t-\tau)}\,z^{(n)}(t-\tau)-Q(t)\,z(t-\sigma)=0. \tag{33}$$

Also

$$\liminf_{t \to \infty} \left[P(t-\sigma) \frac{Q(t)}{Q(t-\sigma)} \right] = p_0.$$

Then for any positive number ε in the interval $0 < \varepsilon < 1/2 \min\{p_0, q_0\}$, Eq. (33) yields the inequality

$$z^{(n)}(t) - (p_0 - \varepsilon) z^{(n)}(t-\tau) - (q_0 - \varepsilon) z(t-\sigma) \geqslant 0.$$

By integrating this inequality from t to ∞ and by using (30) we find

$$-z^{(n-1)}(t) + (p_0 - \varepsilon) z^{(n-1)}(t-\tau) - (q_0 - \varepsilon) \int_t^\infty z(s-\sigma)\, ds \geqslant 0.$$

By repeating the same procedure n times and by noting the fact n is even we are led to the inequality

$$(p_0 - \varepsilon) z(t-\tau) + (q_0 - \varepsilon) \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty z(s-\sigma) \times ds\, ds_1 \cdots ds_{n-1} \leqslant z(t), \qquad t \geqslant T,$$

where T is sufficiently large and $z\colon [T-m, \infty) \to (0, \infty)$, with $m = \max\{\tau, \sigma\}$, is a continuous and strictly decreasing function with limit zero. It follows by Lemma 6 that

$$(p_0 - \varepsilon) v(t-\tau) + (q_0 - \varepsilon) \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty v(s-\sigma) \times ds\, ds_1 \cdots ds_{n-1} = v(t), \qquad t \geqslant T$$

has a continuous bounded positive solution $v\colon [T-m, \infty) \to (0, \infty)$. Clearly $v(t)$ is also a bounded positive solution of the neutral equation

$$\frac{d^n}{dt^n} [v(t) - (p_0 - \varepsilon) v(t-\tau)] - (q_0 - \varepsilon) v(t-\sigma) = 0.$$

Hence, by Lemma 5 and because of the fact that ε is arbitrarily small, it follows that Eq. (28) has a bounded nonoscillatory solution. This contradicts the hypothesis and complete the proof of the theorem. ▮

5. Existence of a Bounded Positive Solution

Consider the neutral delay differential equation

$$\frac{d^n}{dt^n} [y(t) - p(t) y(t-\tau)] - q(t) h(y(t-\sigma)) = 0, \tag{34}$$

where n is an even integer,

$$\begin{aligned} &p, q \in C[[t_0, \infty), \mathbb{R}^+], \qquad \tau \in (0, \infty), \\ &\sigma \in [0, \infty), \qquad \text{and} \qquad h \in C[\mathbb{R}, \mathbb{R}]. \end{aligned} \tag{35}$$

The next theorem is a partial converse of Theorem 2 and shows that, under appropriate hypotheses, Eq. (34) has a bounded positive solution provided that an associated linear equation with constant coefficients has a bounded positive solution.

THEOREM 3. *Assume that* (35) *holds and that there exist positive constants* p_0, q_0, *and* δ *such that*

$$0 \leqslant p(t) \leqslant p_0 \qquad \text{and} \qquad 0 \leqslant q(t) \leqslant q_0 \qquad \text{for} \quad t \geqslant t_0 \tag{36}$$

and that

$$\begin{aligned} &\text{either } 0 \leqslant h(u) \leqslant u \qquad \text{for} \quad 0 \leqslant u \leqslant \delta \\ &\text{or} \qquad 0 \geqslant h(u) \geqslant u \qquad \text{for} \quad -\delta \leqslant u \leqslant 0. \end{aligned} \tag{37}$$

Suppose also that $h(u)$ *is nondecreasing in a neighborhood of the origin and that the characteristic equation of Eq.* (28)

$$\lambda^n - p_0 \lambda^n e^{-\lambda\tau} - q_0 e^{-\lambda\sigma} = 0 \tag{38}$$

has a root in the interval $(-\infty, 0]$. *Then Eq.* (34) *has a bounded non-oscillatory solution.*

Proof. Assume that $0 \leqslant h(u) \leqslant u$ for $0 \leqslant u \leqslant \delta$. The case where $0 \geqslant h(u) \geqslant u$ for $-\delta \leqslant u \leqslant 0$ is similar and will be omitted. Let λ_0 be a root of Eq. (38). As $q_0 > 0$, it follows that $\lambda_0 < 0$. Set $y(t) = \exp(\lambda_0 t)$. Then for T sufficiently large, $0 < y(t) \leqslant \delta$ for $t \geqslant T - \delta$ and h is nondecreasing in $[0, y(T-\delta)]$. Clearly,

$$\begin{aligned} &y(t) > 0, \qquad y'(t) < 0, \\ &y''(t) > 0, ..., y^{(n-1)}(t) < 0, \qquad y^{(n)}(t) > 0, \end{aligned} \tag{39}$$

$$\lim_{t \to \infty} y^{(i)}(t) = 0 \qquad \text{for} \quad i = 0, 1, 2, ..., n \tag{40}$$

and $y(t)$ satisfies the neutral equation

$$\frac{d^n}{dt^n} [y(t) - p_0 y(t-\tau)] - q_0 y(t-\sigma) = 0. \tag{41}$$

By integrating n times both sides of (41) from $t \geqslant T$ to ∞ and by using (40) we obtain

$$y(t) - p_0 y(t-\tau) - \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty q_0 y(s-\sigma)\, ds\, ds_1 \cdots ds_{n-1}$$
$$= 0, \qquad t \geqslant T.$$

In view of (36) and (37), this equation implies that

$$p(t) y(t-\tau) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s-1}^\infty q(s) h(y(s-\sigma))\, ds\, ds_1 \cdots ds_{n-1}$$
$$\leqslant y(t), \qquad t \geqslant T.$$

Then by Lemma 6 it follows that the corresponding equation

$$p(t) x(t-\tau) + \int_t^\infty \int_{s_{n-1}}^\infty \cdots \int_{s_1}^\infty q(s) h(x(s-\sigma))\, ds\, ds_1 \cdots ds_{n-1}$$
$$= x(t), \qquad t \geqslant T$$

has a continuous bounded positive solution $x(t)$. Hence, $x(t)$ is also a solution of Eq. (34). The proof of the theorem is complete. ∎

Finally by combining Theorems 2 and 3 we obtain the following necessary and sufficient condition for the oscillation of every bounded solution of Eq. (34).

COROLLARY. *Assume that* (35) *holds and that there exist* $p_0, q_0 \in (0, \infty)$ *such that*

$$0 < p(t) \leqslant p_0 = \lim_{t \to \infty} p(t) < 1 \qquad \textit{for} \quad t \geqslant t_0$$

and

$$0 \leqslant q(t) \leqslant q_0 = \lim_{t \to \infty} q(t) \qquad \textit{for} \quad t \geqslant t_0.$$

Suppose that

$$\lim_{u \to 0} \frac{h(u)}{u} = 1, \qquad uh(u) > 0 \qquad \textit{for} \quad u \neq 0$$

and that $h(u)$ is nondecreasing in a neighborhood of the origin. Then every bounded solution of Eq. (34) *oscillates if and only if every bounded solution of the linear equation* (27) *oscillates.*

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